Symmetries and classical quantization

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Abstract

A phenomenon of classical quantization is discussed. This is revealed in the class of pseudoclassical gauge systems with nonlinear nilpotent constraints containing some free parameters. Variation of parameters does not change local (gauge) and discrete symmetries of the corresponding systems, but there are some special discrete values of them which give rise to the maximal global symmetries at the classical level. Exactly the same values of the parameters are separated at the quantum level, where, in particular, they are singled out by the requirement of conservation of the discrete symmetries. The phenomenon is observed for the familiar pseudoclassical model of 3D P,T-invariant massive fermion system and for a new pseudoclassical model of 3D P,T-invariant system of topologically massive U(1) gauge fields.

Key words: pseudoclassical gauge systems, quantization, continuous and discrete symmetries, topologically massive gauge fields.

1 Introduction

The quantization of parameters takes place in many physically interesting gauge systems, such as Dirac monopole [1], [2], [3], non-Abelian topologically massive vector gauge theory [4] and its particle-mechanics generalization [5], various spin particle models [6]. In particular, this property is specific to some pseudoclassical spin models [7, 8, 9] belonging to the class of gauge systems with nonlinear nilpotent constraints [7]–[17]. The nature of the quantization phenomenon is hidden in a nontrivial topology of configuration or phase spaces of the corresponding systems.

Here the following interesting question can be formulated. If some system has a quantized parameter, whether its special discrete values may reveal themselves in some way just at the classical level?

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In this letter we shall discuss exactly such a phenomenon, which may be called the classical quantization. We shall show that the requirement of maximality of classical global (rigid) symmetry can separate some special discrete values of corresponding parameters. These values turn out to be exactly the same as those singled out by the quantization procedure. Namely, we shall consider a class of pseudoclassical gauge systems with quadratic in Grassmann variables nilpotent constraints of the form $(\alpha_{ik} + \nu\beta_{ik})\xi_i\xi_k \approx 0$, where ν is a c-valued parameter and α_{ik}, β_{ik} are functions of even variables. While varying the values of the parameter ν, we do not change local continuous and discrete symmetries of the corresponding classical system, but may drastically change its continuous global symmetries. As a result, there are some special discrete values of the parameter, $\nu = \nu_q$, for which a classical system has a maximal global symmetry, whose set of generators includes the integrals of motion existing only at $\nu = \nu_q$. Exactly the same values of the parameter reveal themselves at the quantum level too. But quantum mechanically they are singled out not only by requiring the continuous global symmetry to be maximal. It turns out that only for these special values $\nu = \nu_q$ the discrete symmetries of the corresponding classical system are conserved in the quantum case. So, the classical quantization phenomenon implies also some non-anticipated hidden relationships between continuous and discrete symmetries.

The paper is organized as follows. We start with consideration of the simplest toy model revealing the described phenomenon. Then we investigate in this context the pseudoclassical model [7, 8] of $3D$ P, T-invariant massive fermion system [18] and consider a new pseudoclassical model for $3D$ P, T-invariant system of topologically massive U(1) gauge fields [18]–[20], [4]. We conclude the paper with some remarks.

2 Toy model

Let us consider the following pseudoclassical model given by the Lagrangian

$$L = v\chi + \frac{i}{2}\xi_a\dot{\xi}_a + \frac{i}{2}\theta_a\dot{\theta}_a, \tag{1}$$

where ξ_a, θ_a, $a = 1, 2$, are Grassmann variables, v is an even Lagrange multiplier, and χ is a nilpotent nonlinear function,

$$\chi = -i(\xi_1\xi_2 + \nu\theta_1\theta_2), \tag{2}$$

$\chi^3 = 0$, containing a real c-number parameter ν. When introducing the notation $\xi^\pm = \frac{1}{\sqrt{2}}(\xi_1 \pm i\xi_2)$, $\theta^\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$, one can present χ in the form $\chi = \xi^+\xi^- + \nu\theta^+\theta^-$.

Lagrangian (1) leads to the nontrivial Dirac brackets $\{\xi^+, \xi^-\} = -i$, $\{\theta^+, \theta^-\} = -i$ and $\{v, p_v\} = 1$, and generates the first class primary, $p_v \approx 0$, and secondary, $\chi \approx 0$, constraints [21]. The total Hamiltonian is a linear combination of the constraints, $H = v\chi + up_v$, with $u = u(\tau)$ being an arbitrary function of the evolution parameter. The phase space variables satisfy the following Hamiltonian equations of motion: $\dot{p}_v = 0$, $\dot{v} = u$, $\dot{\xi}^\pm = \pm iv\xi^\pm$, $\dot{\theta}^\pm = \pm i\nu v\theta^\pm$. Their general solutions are $p_v(\tau) = p_v(0)$, $v(\tau) = v(0) + \int_0^\tau u(\tau')d\tau'$, $\xi^\pm(\tau) = e^{\pm i\omega(\tau)}\xi^\pm(0)$, $\theta^\pm(\tau) = e^{\pm i\nu\omega(\tau)}\theta^\pm(0)$, where $\omega(\tau) = \int_0^\tau v(\tau')d\tau'$. One finds the obvious integrals of motion, $\mathcal{N}_\xi = \xi^+\xi^-$, $\mathcal{N}_\theta = \theta^+\theta^-$. Therefore, our nilpotent constraint is their linear combination, $\chi = \mathcal{N}_\xi + \nu\mathcal{N}_\theta$. The case of $\nu = 0$ is degenerate: in this case variables $\theta^\pm$ have no dynamics, $\theta^\pm(\tau) = \theta^\pm(0)$, being trivial integrals of motion. Now let us put the question: are there

other values of the parameter ν which would be special from the point of view of dynamics? Using the explicit solution to the equations of motion, one finds that such special values are $\nu = \pm 1$. Only in these cases there are two additional integrals of motion,

$$\mathcal{T}_+^+ = \xi^+ \theta^-, \quad \mathcal{T}_+^- = \theta^+ \xi^- = (\mathcal{T}_+^+)^* \tag{3}$$

for $\nu = +1$, and $\mathcal{T}_-^+ = \xi^+ \theta^+$, $\mathcal{T}_-^- = \theta^- \xi^- = (\mathcal{T}_-^+)^*$ for $\nu = -1$. Via the trivial change of the variables, $\theta^\pm \to \theta^\mp$, the second case can be reduced to the first one. Note that when $\nu \neq 1$, one can construct the integrals $\mathcal{T}_\nu^+ = \xi^+(\tau)\theta^-(\tau)e^{i(\nu-1)\omega(\tau)}$, $\mathcal{T}_\nu^- = (\mathcal{T}_\nu^+)^*$, which are *nonlocal in the evolution parameter τ functions* due to the presence of the factor $e^{i(\nu-1)\omega(\tau)}$. They become local integrals (3) at $\nu = 1$. In the same way one can construct nonlocal integrals which turn into the local integrals $\mathcal{T}_-^+$, $\mathcal{T}_-^-$ at $\nu = -1$. Thus, we conclude that there are special values of c-number parameter ν, $\nu = \pm 1$, at which the system has additional nontrivial local integrals of motion and so, these cases can be singled out by requiring the maximal continuous global symmetry in the system. This continuous global symmetry is generated by the integrals $\mathcal{N}_\xi$, $\mathcal{N}_\theta$ and the additional integrals $\mathcal{T}_+^\pm$ or $\mathcal{T}_-^\pm$.

To conclude the discussion of the classical theory, one notes that the system (1) is invariant also with respect to the discrete transformation

$$D : \xi_a \to (\xi_1, -\xi_2), \quad D : \theta_a \to (\theta_1, -\theta_2), \quad D : v \to -v, \tag{4}$$

taking place for the arbitrary value of the parameter ν. As we shall see, the quantum analog of this discrete symmetry plays very important role.

In correspondence with classical brackets, at the quantum level we have $[\widehat{\xi}_a, \widehat{\xi}_b]_+ = \delta_{ab}$, $[\widehat{\theta}_a, \widehat{\theta}_b]_+ = \delta_{ab}$, $[\widehat{\xi}_a, \widehat{\theta}_b]_+ = 0$. One can realize $\widehat{\xi}_a$ and $\widehat{\theta}_a$ as the operators $\widehat{\xi}_a = \frac{1}{\sqrt{2}}\sigma_a \otimes \sigma_3$, $\widehat{\theta}_a = \frac{1}{\sqrt{2}}1 \otimes \sigma_a$ acting on the space of functions $\Psi = \psi_1 \otimes \psi_2$, where $\psi_{1,2} = \psi_{1,2}(v)$ are two-component functions. Then the quantum analog of the constraint $p_v \approx 0$, $\widehat{p}_v \Psi = 0$ with $\widehat{p}_v = -i\partial/\partial v$, means that the function Ψ does not depend on v. Finally, the physical subspace of the system is singled out by the quantum analog of the second constraint, $\widehat{\chi}\Psi = 0$. There is an operator-ordering ambiguity under construction of the quantum analog of the nonlinear nilpotent constraint, and in general case we have

$$\widehat{\chi} = \sigma_3 \otimes 1 + \nu \cdot 1 \otimes \sigma_3 + \alpha \tag{5}$$

with the constant α (of order $\hbar$) characterizing the deviation of the ordering from that one corresponding to the classical ordering in (2), i.e. $\alpha = 0$ corresponds to the antisymmetrized ordering of operators $\widehat{\xi}^+$, $\widehat{\xi}^-$ and $\widehat{\theta}^+$, $\widehat{\theta}^-$ (see ref. [7] for more detailed discussion of this point). In the case of antisymmetrized ordering ($\alpha = 0$), we find that the quantum constraint $\widehat{\chi}\Psi = 0$ has nontrivial solutions iff $|\nu| = 1$. The corresponding physical states in transposed form are given by $\Psi_{+,1}^t = (1,0)\otimes(0,1)$ and $\Psi_{+,2}^t = (0,1)\otimes(1,0)$ for $\nu = +1$, and $\Psi_{-,1}^t = (1,0)\otimes(1,0)$, $\Psi_{-,2}^t = (0,1) \otimes (0,1)$ for $\nu = -1$. Therefore, we see that exactly the same values of the parameter ν turn out to be special from the quantum point of view. This is, of course, not an accidental fact. Indeed, e.g., in the case $\nu = +1$ the physical states $\Psi_{+,1}$, $\Psi_{+,2}$ are the eigenstates of the operators $\widehat{\mathcal{N}}_\xi = \frac{1}{2}(\sigma_3 + 1) \otimes 1$, $\widehat{\mathcal{N}}_\theta = 1 \otimes \frac{1}{2}(\sigma_3 + 1)$, whereas they are transformed mutually by the operators $\widehat{\mathcal{T}}_+^+ = -\sigma_+ \otimes \sigma_-$ and $\widehat{\mathcal{T}}_+^- = -\sigma_- \otimes \sigma_+$. Therefore, the physical operators $\widehat{\mathcal{N}}_\xi$, $\widehat{\mathcal{N}}_\theta$, $\widehat{\mathcal{T}}_+^+$ and $\widehat{\mathcal{T}}_+^-$ can be considered as generators of the corresponding global (i.e. constant in τ) symmetry.

At the classical level we had another, dynamically degenerated special case corresponding to $\nu = 0$. So, let us put $\nu = 0$ and consider the quantum constraint $\widehat{\chi}$ in the most general form (5) containing ordering parameter α. Then one can find that the quantum condition $\widehat{\chi}\Psi = 0$ has nontrivial solutions only either at $\alpha = 1$ that corresponds to the normal ordering of the operators $\widehat{\xi}^{+}$, $\widehat{\xi}^{-}$ under construction of the quantum analog of our basic nilpotent constraint, $\widehat{\chi} = \widehat{\xi}^{+}\widehat{\xi}^{-}$, or at $\alpha = -1$ in the case of antinormal ordering. In these cases, in correspondence with classical picture, the operators $\widehat{\theta}^{\pm}$ will be physical operators additional to the physical operator $\widehat{\mathcal{N}}_{\xi}$.

Let us analyze the general quantum case given by arbitrary values of both parameters ν and α. One finds that if $|\nu| \neq 0, 1$, the quantum constraint $\widehat{\chi}\Psi = 0$ has nontrivial solution under appropriate choice of the parameter α (in this case the corresponding value of this parameter is different from ± 1 and 0), however there is *only one* corresponding physical state annihilated by $\widehat{\chi}$. Therefore, the same values of the parameter ν ($\nu = \pm 1$, 0) turn out to be special also from the point of view of the quantum theory if we require the maximal number of solutions of the quantum constraint $\widehat{\chi}\Psi = 0$, so that the maximal (global) symmetry would be realized on the corresponding physical states.

Let us recall that we had also the classical discrete symmetry (4). What does happen with it upon quantization? The corresponding unitary operator generating this discrete symmetry is $U_D = R_v \sigma_2 \otimes \sigma_1$, where the reflection operator R_v is given by the relations $R_v^2 = 1$, $R_v v = -v R_v$. Operator U_D transforms the state $\Psi_{+,1}$ into $\Psi_{+,2}$ and $\Psi_{-,1}$ into $\Psi_{-,2}$, and vice versa. Therefore, in accordance with the preceding discussion, the discrete symmetry (4) survives at the quantum level only when the parameter takes the special values $\nu = \pm 1$ (and when, therefore, the ordering parameter α is zero).

Further we shall see that in two concrete physical models of P, T-invariant $3D$ systems the special values of the corresponding parameters can also be separated at the quantum level *by requiring the conservation of the discrete symmetries taking place at the classical level.* Exactly the same special discrete values will be singled out classically and quantum mechanically by requiring the maximality of the global symmetries, as it just happened in the toy model.

3 $3D$ massive double fermion system

Let us consider first the pseudoclassical physical model of $3D$ P, T-invariant massive fermion system revealing the structure similar to that of the toy model. Such a model is given by the Lagrangian [7, 8]

$$L = \frac{1}{2e}(\dot{x}_\mu - iv\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda)^2 - \frac{e}{2}m^2 + 2i\nu m v\theta_1\theta_2 - \frac{i}{2}\xi_\mu\dot{\xi}^\mu + \frac{i}{2}\theta_a\dot{\theta}_a, \qquad (6)$$

where θ_a, $a = 1, 2$, are Grassmann scalar variables, ξ_μ, $\mu = 0, 1, 2$, is Grassmann Lorentz vector, e and v are even Lagrange multipliers, x_μ are space-time coordinates of the particle, m is a mass parameter, ν is the c-number parameter, and we use the metric $\eta_{\mu\nu} = diag(-, +, +,)$ and the totally antisymmetric tensor $\epsilon_{\mu\nu\lambda}$, $\epsilon^{012} = 1$. Lagrangian (6) is invariant with respect to the discrete P and T transformations [8] (cp. with ref. [22], where the time-reversal symmetry of the relativistic spinning particle is analyzed), $P : X_\mu \rightarrow (X_0, -X_1, X_2)$, $T :$

$X_\mu \to (-X_0, X_1, X_2)$, where $X_\mu = x_\mu, \xi_\mu$, and $P :, T : (e, v) \to (e, -v)$ $P : \theta_a \to (\theta_1, -\theta_2)$, $T : \theta_a \to (-\theta_1, \theta_2)$. It is necessary to stress that the invariance with respect to these discrete transformations is valid *classically* for any value of the parameter ν.

The Hamiltonian description of the system is given by nontrivial brackets $\{\theta_a, \theta_b\} = -i\delta_{ab}$, $\{\xi_\mu, \xi_\nu\} = i\eta_{\mu\nu}$, $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$, $\{e, p_e\} = 1$, $\{v, p_v\} = 1$, by the set of the first class primary, $p_e \approx 0$, $p_v \approx 0$, and secondary,

$$\phi = \frac{1}{2}(p^2 + m^2) \approx 0, \quad \chi = i(\epsilon_{\mu\nu\lambda}p^\mu\xi^\nu\xi^\lambda - 2\nu m\theta_1\theta_2) \approx 0,$$

constraints. The total Hamiltonian $H = e\phi + v\chi + u_1 p_e + u_2 p_v$ contains arbitrary functions $u_{1,2} = u_{1,2}(\tau)$ [21] and generates the following equations of motion: $\dot{p}_\mu = \dot{p}_e = \dot{p}_v = 0$, $\dot{x}_\mu = ep_\mu + i\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda$, $\dot{\xi}_\mu = v\epsilon_{\mu\nu\lambda}p^\nu\xi^\lambda$, $\dot{\theta}^\pm = \pm i\nu m v\theta^\pm$, $\dot{e} = u_1$, $\dot{v} = u_2$. One can immediately identify the essential integrals of motion: the energy-momentum vector p_μ, the total angular momentum vector $J_\mu = -\epsilon_{\mu\nu\lambda}x^\nu p^\lambda - \frac{i}{2}\epsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda$, and $\Gamma = p\xi$, $\mathcal{N}_\theta = \theta^+\theta^-$, $\Delta = i\epsilon_{\mu\nu\lambda}p^\mu\xi^\nu\xi^\lambda$. Therefore, the nilpotent constraint is again a linear combination of the integrals of motion, $\chi = \Delta + 2\nu m\mathcal{N}_\theta$. The scalar Grassmann variables $\theta^\pm$ have the dynamics of the type we had in the toy model, $\theta^\pm(\tau) = e^{\pm i\nu\omega(\tau)}\theta^\pm(0)$, where now $\omega(\tau) = m\int_0^\tau v(\tau')d\tau'$.

Using the mass shell constraint $\phi \approx 0$, one may introduce the complete oriented triad $e_\mu^{(\alpha)} = e_\mu^{(\alpha)}(p)$, $\alpha = 0, 1, 2$, $e_\mu^{(0)} = p_\mu/\sqrt{-p^2}$, $e_\mu^{(\alpha)}\eta_{\alpha\beta}e_\nu^{(\beta)} = \eta_{\mu\nu}$, $\epsilon_{\mu\nu\lambda}e^{(0)\mu}e^{(i)\nu}e^{(j)\lambda} = \epsilon^{0ij}$. When defining $\xi^{(\alpha)} = \xi^\mu e_\mu^{(\alpha)}$ and $\xi^{(\pm)} = \frac{1}{\sqrt{2}}(\xi^{(1)} \pm i\xi^{(2)})$, one has $\{\xi^{(+)}, \xi^{(-)}\} = i$, which differs in sign from the brackets for the variables $\theta^\pm$. It is necessary to note that the space-like components of the triad $e_\mu^{(i)}$, $i = 1, 2$, are not Lorentz vectors (see, e.g. ref. [6]), and so, the quantities $\xi^{(i)}$ as well as $\xi^{(\pm)}$ are not Lorentz scalars. With the help of the mass shell constraint, the nilpotent constraint can be presented in the equivalent form $\chi = 2m(\mathcal{N}_\xi - \nu\mathcal{N}_\theta)$, where $\mathcal{N}_\xi = \xi^{(+)}\xi^{(-)}$ is the integral of motion coinciding up to the sign with the spin, $\mathcal{N}_\xi = -J^{(0)}$. The variables $\xi^{(\pm)}$ have the evolution law analogous to that in the toy model: $\xi^{(\pm)}(\tau) = e^{\pm i\omega(\tau)}\xi^{(\pm)}(0)$. Therefore, exactly as in the case of the toy model, we have additional integrals of motion $\mathcal{T}_+^\pm$ or $\mathcal{T}_-^\pm$ if and only if $\nu = +1$ or -1, and they are given by the same expressions (with the change $\xi^\pm \to \xi^{(\pm)}$) presented in the toy model. Of course, the case $\nu = 0$ is again degenerated with variables $\theta^\pm$ being trivial integrals of motion. As we shall see, this case is completely excluded at the quantum level. Hence, we conclude that at the classical level the discrete values $\nu = \pm 1$ are special due to exactly the same reasons we outlined in the case of the toy model.

Following the classical brackets, quantum operators associated with the odd variables ξ^μ can be realized as $\hat{\xi}^\mu = \frac{1}{\sqrt{2}}\gamma^\mu \otimes \sigma_3$, whereas operators $\hat{\theta}_a$ can be realized as $\hat{\theta}_a = \frac{1}{\sqrt{2}}1 \otimes \sigma_a$. Here the γ-matrices satisfy the relation $\gamma_\mu\gamma_\nu = -\eta_{\mu\nu} + i\epsilon_{\mu\nu\lambda}\gamma^\lambda$ and can be chosen in the form $\gamma^0 = \sigma_3$, $\gamma^i = i\sigma_i$, $i = 1, 2$. It is convenient to assume that the quantum states Ψ in transposed form are presented as $\Psi^t = (\psi_u^t, \psi_d^t)$, and that γ-matrices act on the spinor indices of the states ψ_u and ψ_d, whereas σ-matrices of the second factor in the operators $\hat{\xi}_\mu$ and $\hat{\theta}_a$ act in the space specified by the indices u and d. Quantum constraints $\hat{p}_e\Psi = \hat{p}_v\Psi = 0$ with $\hat{p}_e = -i\partial/\partial e$, $\hat{p}_v = -i\partial/\partial v$ mean the independence of physical states from e and v, and the essential quantum conditions are

$$\hat{\phi}\Psi = 0, \quad \hat{\chi}\Psi = 0. \tag{7}$$

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Here the quantum counterpart of the nilpotent constraint has the form $\widehat{\chi} = \widehat{p}_\mu \gamma^\mu \otimes 1 + \nu m (1 \otimes \sigma_3 + \alpha)$, where $\alpha = 0$ corresponds to the antisymmetrized ordering of the operators $\widehat{\theta}^+$, $\widehat{\theta}^-$. In this case ($\alpha = 0$), one can easily check that two quantum conditions (7) have nontrivial solutions iff $\nu = \epsilon = \pm 1$. The corresponding physical states ψ_u and ψ_d form the pair of (2+1)-dimensional Dirac fields, $(-i\partial_\mu \gamma^\mu + \epsilon m)\psi_u = 0$, $(-i\partial_\mu \gamma^\mu - \epsilon m)\psi_d = 0$. Therefore, in both cases, $\nu = +1$ and $\nu = -1$, we have a P, T-invariant system of two fermion fields with opposite spins $+1/2$ and $-1/2$.

Let us assume now that $\alpha \neq 0$. This corresponds to the operator ordering in $\widehat{\chi}$ different from the antisymmetrized ordering. Then we find that at $|\nu| \neq 1$ quantum conditions (7) have nontrivial solutions when the values of α and ν are related as $(\nu(\alpha + 1))^2 = 1$ or $(\nu(\alpha - 1))^2 = 1$. In these cases we have as the physical state only one of the two, ψ_u or ψ_d, satisfying the corresponding Dirac equation, but not the both states. This means that at $|\nu| \neq 1$ only one fermion state of spin $+1/2$ or $-1/2$ will be physical, and so, the P, T-invariance of the classical theory will be broken [8]. Moreover, we see that $\nu = 0$ is completely excluded quantum mechanically since the set of equations (7) has no nontrivial solutions in this case.

We conclude that in the case of the pseudoclassical model given by the Lagrangian (6), the same discrete values of the parameter, $\nu = \pm 1$, turn out to be special from the classical and the quantum points of view. At $|\nu| = 1$ the quantum analogs of the corresponding additional integrals of motion together with quantum analogs of integrals $\mathcal{N}_\xi$, $\mathcal{N}_\theta$ generate U(1,1) dynamical symmetry and $N = 3$ supersymmetry, and as it was shown in ref. [8], the physical states realize irreducible representation of a nonstandard superextension of the (2+1)-dimensional Poincaré group.

4 P, T-invariant system of topologically massive U(1) gauge fields

Let us consider the pseudoclassical model of P, T-invariant system of topologically massive U(1) gauge fields. The novel feature which will be revealed here is that the corresponding additional integrals of motion taking place at special values of the corresponding c-number parameter ν will be of the third order in Grassmann variables though the corresponding nilpotent constraint will again be quadratic.

The model is given by the Lagrangian

$$L = \frac{1}{2e}\left(\dot{x}_\mu - \frac{i}{2}\upsilon\epsilon_{\mu\nu\lambda}\xi_a^\nu\xi_a^\lambda\right)^2 - \frac{e}{2}m^2 - i\nu m\upsilon\xi_1^\mu\xi_{2\mu} - \frac{i}{2}\xi_{a\mu}\dot{\xi}_a^\mu, \tag{8}$$

where now we have two Grassmann Lorentz vectors, ξ_a^μ, $a = 1, 2$, instead of one vector ξ_μ and two scalar variables θ_a from the previous model. The model is invariant with respect to P and T transformations, under which the variables x_μ, e and υ are transformed in the same way as in the previous model, ξ_1^μ is transformed as ξ^μ, while for ξ_2^μ we have additional sign factor in comparison with the transformation of ξ_1^μ, $P : \xi_2^\mu \to -(\xi_2^0, -\xi_2^1, \xi_2^2)$, $T : \xi_2^\mu \to -(-\xi_2^0, \xi_2^1, \xi_2^2)$, i.e. we suppose that ξ_2^μ is a pseudovector. Again, classically P- and T-invariance take place for arbitrary values of the parameter ν.

Lagrangian (8) generates the constraints of the same form as we had in the fermion model, with the only difference in the structure of the nilpotent constraint. Now it is given by

$$\chi = \frac{i}{2}(\epsilon_{\mu\nu\lambda}p^{\mu}\xi_{a}^{\nu}\xi_{a}^{\lambda} + 2\nu m\xi_{1}\xi_{2}) \approx 0. \tag{9}$$

The obvious essential integrals of motion are energy-momentum vector p_{μ} and the total angular momentum vector $J_{\mu} = -\epsilon_{\mu\nu\lambda}x^{\nu}p^{\lambda} - \frac{i}{2}\epsilon_{\mu\nu\lambda}\xi_{a}^{\nu}\xi_{a}^{\lambda}$.

It is convenient to introduce the complex mutually conjugate vector variables, $b_{\mu}^{\pm} = \frac{1}{\sqrt{2}}(\xi_{1\mu} \pm i\xi_{2\mu})$. They have the following brackets: $\{b_{\mu}^{+}, b_{\nu}^{-}\} = i\eta_{\mu\nu}$, $\{b_{\mu}^{+}, b_{\nu}^{+}\} = \{b_{\mu}^{-}, b_{\nu}^{-}\} = 0$. Equations of motion for these variables have the form

$$\dot{b}_{\mu}^{\pm} = v(\epsilon_{\mu\nu\lambda}p^{\nu} \pm i\nu m\eta_{\mu\lambda})b^{\pm\lambda}. \tag{10}$$

Using the triad $e_{\mu}^{(\alpha)}(p)$ and notations $b^{(\alpha)\pm} = b_{\mu}^{\pm}e^{(\alpha)\mu}$, we find the general solution to the equations of motion for odd variables

$$b^{(0)\pm}(\tau) = e^{\pm i\nu\omega(\tau)}b^{(0)\pm}(0), \quad b^{(i)\pm}(\tau) = e^{\pm i\nu\omega(\tau)}\left(\cos\omega(\tau)b^{(i)\pm}(0) - \epsilon^{ij}\sin\omega(\tau)b^{(j)\pm}(0)\right),$$

where $\epsilon^{ij} = \epsilon^{0ij}$, $\omega(\tau) = m\int_{0}^{\tau}v(\tau')d\tau'$. From here we get the quadratic in odd variables integrals of motion, $-i\epsilon^{ij}b^{(i)+}b^{(j)-} = J^{(0)}$ (spin of the system), and $\mathcal{N}_{0} = b^{(0)+}b^{(0)-}$, $\mathcal{N}_{\perp} = b^{(i)+}b^{(i)-}$. With the help of the mass shell constraint, the nilpotent constraint is presented as a linear combination of the integrals, $\chi = -m(J^{(0)} + \nu(-\mathcal{N}_{0} + \mathcal{N}_{\perp}))$. Again, we find that the case $\nu = 0$ is special: it gives dynamically trivial variables $b^{(0)\pm}$, $b^{(0)\pm}(\tau) = b^{(0)\pm}(0)$, and as we shall see, this value of ν will be excluded at the quantum level. In order to reveal nontrivial special values of the parameter, let us construct the following quadratic in odd variables combinations: $A^{\pm} = (b^{(2)+}b^{(2)-} - b^{(1)+}b^{(1)-}) \pm i(b^{(2)+}b^{(1)-} + b^{(1)+}b^{(2)-})$. They satisfy a simple evolution law: $A^{\pm}(\tau) = e^{\mp 2i\omega(\tau)}A^{\pm}(0)$. We immediately conclude that when $|\nu| = 2$, there are two additional integrals of motion of the third order in odd variables. They are $\mathcal{A}_{+}^{+} = A^{+}b^{(0)+}$, $\mathcal{A}_{+}^{-} = (\mathcal{A}_{+}^{+})^{*}$ for $\nu = +2$ and $\mathcal{A}_{-}^{+} = A^{+}b^{(0)-}$, $\mathcal{A}_{-}^{-} = (\mathcal{A}_{-}^{+})^{*}$ for $\nu = -2$, and so, this model reveals the classical quantization of the parameter.

Let us quantize the model. In correspondence with classical brackets, the quantum counterparts of the odd variables $b_{\mu}^{\pm}$ have to form the algebra of fermionic creation-annihilation operators, $[\hat{b}_{\mu}^{+}, \hat{b}_{\nu}^{-}]_{+} = -\eta_{\mu\nu}$, $[\hat{b}_{\mu}^{+}, \hat{b}_{\nu}^{+}]_{+} = [\hat{b}_{\mu}^{-}, \hat{b}_{\nu}^{-}]_{+} = 0$. After taking into account the quantum analogs of the constraints $p_{e} \approx 0$, $p_{v} \approx 0$, two remaining quantum constraints are to annihilate the states of the general form

$$\Psi(x) = \left(f(x) + \mathcal{F}_{-}^{\mu}(x)\hat{b}_{\mu}^{+} + \epsilon_{\mu\nu\lambda}\mathcal{F}_{+}^{\mu}(x)\hat{b}^{+\nu}\hat{b}^{+\lambda} + \tilde{f}(x)\epsilon_{\mu\nu\lambda}\hat{b}^{+\mu}\hat{b}^{+\nu}\hat{b}^{+\lambda}\right)|0\rangle,$$

with the vacuum state $|0\rangle$ defined by $\hat{b}_{\mu}^{-}|0\rangle = 0$. The quantum P and T transformations are given here as $P, T : \Psi(x) \to \Psi'(x'_{P,T}) = U_{P,T}\Psi(x)$, $x_{P}'^{\mu} = (x^{0}, -x^{1}, x^{2})$, $x_{T}'^{\mu} = (-x^{0}, x^{1}, x^{2})$, where $U_{P} = B_{+}^{0}B_{-}^{1}B_{+}^{2}$, $U_{T} = B_{-}^{0}B_{+}^{1}B_{+}^{2}$, $B_{\pm}^{\mu} = \hat{b}^{+\mu} \pm \hat{b}^{-\mu}$. Operators U_{P} and U_{T} are antiunitary, and in correspondence with classical relations, they give $U_{P}\hat{b}_{0,2}^{\pm}U_{P}^{-1} = \hat{b}_{0,2}^{\mp}$, $U_{P}\hat{b}_{1}^{\pm}U_{P}^{-1} = -\hat{b}_{1}^{\mp}$, $U_{T}\hat{b}_{1,2}^{\pm}U_{T}^{-1} = \hat{b}_{1,2}^{\mp}$, $U_{T}\hat{b}_{0}^{\pm}U_{T}^{-1} = -\hat{b}_{0}^{\mp}$. One can easily check that while acting on the field $\Psi(x)$, these operators transform scalar fields $f(x)$ and $\tilde{f}(x)$ one into another, and induce mutual transformation of the vector fields $\mathcal{F}_{+}^{\mu}(x)$ and $\mathcal{F}_{-}^{\mu}(x)$.

The physical states are singled out by the quantum analogs of remaining two first class constraints, $(-\partial^2 + m^2)\Psi = 0$, $\widehat{\chi}\Psi = 0$. Let us fix the same ordering in the quantum operator $\widehat{\chi}$ as in the classical constraint (9). This gives $\widehat{\chi} = \epsilon_{\mu\nu\lambda}\widehat{b}^{+\mu}\widehat{b}^{-\nu}\partial^\lambda - \nu m(\widehat{b}^+_\mu\widehat{b}^{-\mu} + 3/2)$. As a consequence of the quantum constraints, we immediately find that $f(x) = \tilde{f}(x) = 0$, whereas for the fields $\mathcal{F}^\mu_\pm$ we have the equations

$$\left(\epsilon_{\mu\nu\lambda}\partial^\nu \mp \frac{1}{2}\nu m \eta_{\mu\lambda}\right)\mathcal{F}^\lambda_\pm = 0, \tag{11}$$

and $(-\partial^2 + m^2)\mathcal{F}^\mu_\pm = 0$. It is interesting to note that equations (11) contain the tensor operator being the quantum counterpart of the classical tensor quantity generating the classical equations of motion (10) for $b^\pm_\mu$. Due to the linear equations (11), we have also $\partial_\mu \mathcal{F}^\mu_\pm = 0$ and $(-\partial^2 + \frac{1}{4}\nu^2 m^2)\mathcal{F}^\mu_\pm = 0$. Hence, the quantum constraints are consistent (i.e. have nontrivial solutions) if and only if $|\nu| = 2$, i.e. we arrive at the same quantization condition which we have obtained at the classical level. Putting $\nu = 2\epsilon$, $\epsilon = +$ or $-$, we get finally that the quantum analog of the nilpotent constraint gives us the equations for P, T-invariant system of topologically massive U(1) gauge fields carrying spins $+1$ and -1 [18].

If we choose another ordering prescription under construction of the quantum analog of the constraint χ, we would have the same quantum operator but with the constant term $3/2$ changed for $3/2 + \alpha$, where constant α specifies the ordering. As a result, under appropriate choice of the parameter ν, $|\nu| \neq 2, 0$, for $\alpha \neq 0, -3/2, +3/2$ we would have, as a solution of the quantum constraints, only one field $\mathcal{F}^\mu_+$ or $\mathcal{F}^\mu_-$ satisfying the corresponding linear differential equation. In this case we would have violation of the discrete P, T-symmetries taking place in the system at the classical level.

In the cases $\alpha = -3/2$ (or $\nu = 0$) or $\alpha = +3/2$, the physical states would be given by one scalar field $f(x)$ or $\tilde{f}(x)$, respectively, and in correspondence with the discussion above, for both these cases we again lose P- and T-invariance.

Therefore, the same values of the parameter ν, $\nu = \pm 2$, which we have separated classically, turn out to be special quantum mechanically: in these cases the number of physical states is maximal, and only at $\nu = \pm 2$ we conserve the P, T-symmetries.

One can show [23] that the quantum counterparts of the additional integrals of motion $\mathcal{A}^\pm_+$ or $\mathcal{A}^\pm_-$ give rise to the hidden U(1,1) symmetry and $N = 3$ supersymmetry, as it takes place for the P, T-invariant massive fermionic system.

5   Concluding remarks

We have revealed a phenomenon of the classical quantization for the particular class of pseudoclassical systems containing nilpotent quadratic in Grassmann variables constraints [7]–[17]. The peculiarity generic to this class of constraints is that they admit no, even local, gauge conditions [17]. Following the ideas of the present paper it would be interesting to investigate other known pseudoclassical spin particle models [9]-[16] belonging to this class. It seems very likely that the same phenomenon can also be revealed in pseudoclassical systems with higher (than second) order nilpotent constraints belonging to the same peculiar class of constraints [17].

One might try to generalize the class of models we have considered to the case of systems with infinite number of odd degrees of freedom. One could this way arrive at some interesting from the physical point of view spin chain systems revealing the phenomenon of classical quantization.

Concluding, it is of interest to investigate in the same context non-Grassmannian mechanical [6] and field [1, 4] gauge systems with quantized parameters.

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